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A Mathematical Model for the Dynamics of Information Spread under the Effect of Social Response

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Mark Granovetter promoted the threshold model of social behavior in which the acceptance value of an action is determined by the proportion of a population that already accepted it. The model is about an individual embracing an idea once a sufficient number of people embrace it. In this paper, we propose a mathematically accurate population dynamics model based on Granovetter's idea for the spread of information in a population. Individual threshold values with respect to the acceptance of a piece of information are distributed throughout the population ranging from low (easily accepts information) to high (hardly accepts). Results from the mathematical analysis on our model show that critical values exist for *initial knower population size*, *mean* and *variance* of threshold values. These critical values are about the drastic difference in the proportion of the population that end up knowing the information, depending on respective features of the population according to the information spread.

KEYWORDS: information spread, threshold model, collective behavior, population dynamics, differential equations

1. Introduction

Granovetter [1] developed the models of collective behavior for circumstances where people have to make one of two clearly different choices such that the merit or demerit of each choice depends on the number of individuals who decide for or against it. When this number of individuals (which constitute a threshold) is reached, the advantages of taking the decision begin to outweigh the disadvantages for a given individual. For instance, a radical who is capable of single-handedly starting a riot can be said to have a threshold of 0% as they are able to riot even if nobody else toes that line. On the other hand, a conservative might have a threshold close to 100% depending on their level of reluctance to join a riot. The principle of threshold is analogous to credulity and vulnerability in the spread of rumors and diseases respectively. For some background work on this idea, see [2, 3].

Granovetter & Soong [4] emphasized the reality of complex heterogeneity in collective behavior as opposed to the earlier simplifying assumptions of homogeneous individuality and mixing in the adoption and spread of ideas. They showed the importance of threshold models as lying in the not-so-simple connection between individual choices and overall steady results. The work also refers to the importance of bandwagon effect in which people adopt a new concept because a given number of people are into it and a snob effect in which some people drop the idea once a certain number of people sign up. In this case, there are two threshold values: one minimum inspiring the bandwagon and one maximum leading to snobbish behavior.

As a sociological concept, the Granovetter model has some similarities with the idea of behavioral contagion in psychology and the cultural phenomenon of bandwagon effect. In order to make sense of the concept of social influence, Dodds & Watts [5] reasoned that it can be viewed as a result of making decisions based on a series of binary possibilities. These threshold models find application in areas like diffusion of innovation, public protests, migration, voting, market trends, international relations and information spread (see [6–10]).

Various mathematical and computational approaches abound in literature for studying collective behavior. Castellano *et al.* [11] highlights the relevance of statistical physics to other areas of learning apart from physics. The application of concepts in the field to the study of collective behavior in social systems was seen to be fast emerging. Assuming a network that is random and non-finite with weak connections, Whitney [12] tried to understand diffusion

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(of information or innovation) on the network using generating functions. The theory proposed is based on a threshold rule which ensures that a node only changes state after a fraction of nearby nodes, surpassing a particular limit, have previously flipped over. Akhmetzhanov *et al.* [13] extended the Granovetter model to consider a network of individuals in a square lattice with each one having a state and a specified threshold for change in behavior. A utility-psychological threshold model based on the Granovetter's threshold model was introduced by Li & Tang [14]. They studied the critical shift in phase of group behavior by taking into account rational utility and psychological thresholds under the influence of space and intensity of social network. We find other interesting methods in [15–23].

Previous models [1, 4, 6, 14, 24] described the conceptual process of what is called Granovetter's threshold model which, however, cannot be regarded as a population dynamics model for the temporal variation of the number/proportion of "information knowers." In this paper, we construct a mathematically accurate population dynamics model with reasonable assumptions which correspond with the assumptions for Granovetter's threshold model. The derived model is described by a novel formulation which needs to be mathematically investigated in terms of the nature of the population dynamics governed by it. Our analyses on the population dynamics model show that its mathematical features are qualitatively analogous to those from previous models, whereas their results could not be regarded as ones derived for the temporal variation of population dynamics.

A reasonable population dynamics model for the Granovetter's threshold model is worth deriving because it could become a basic model for theoretical consideration in a variety of information spread contexts. It is necessary to have a reasonable population dynamics model in order to discuss the nature of temporal change in social situations with temporal variations in their subpopulation sizes. So, our model is expected to provide a basic population dynamics model for such a problem.

2. Modeling the Population Dynamics of Information Spread

2.1 Assumptions

We have the following underlying assumptions for our modeling with some appropriate generalization of Granovetter's idea [1–3]:

(1) There is a piece of information spreading within a population with a given strength of social recognition effect Q. The social recognition effect represents the acceptability/attractiveness of the information;
(2) The strength of social recognition effect Q increases with the proportion/frequency P of knowers of the information because an increasing number of people are embracing it;
(3) Each individual has a threshold value ξ according to the strength of social recognition effect Q. It determines whether the information is accepted or ignored by that individual;
(4) The threshold value ξ, which characterizes each individual, is constant independently of time and social situation. This means that each person's attitude towards the information is fixed over time no matter what happens;
(5) Every information knower contributes to the social recognition effect at any time t. The contribution could appear as the transmission of the information to others;
(6) Information knowers never return to being non-knowers. That is, there is no forgetfulness.

The assumptions 5 and 6 mean that we consider the population dynamics in a time scale in which the information is maintained in the population and remains in circulation subject to the social recognition effect.

2.2 Mathematical setup

Let us assume the strength of social recognition effect $Q = Q(P)$ as a function of the frequency P of knowers in the population. It is non-decreasing in terms of P, with $Q(0) = 0$ and $Q(P) \geq 0$ for $P \geq 0$. The threshold value ξ according to Q specifies the individual independently of time:

$$\begin{cases} \xi \leq Q \rightarrow \text{The individual may accept the information;} \\ \xi > Q \rightarrow \text{The individual ignores the information.} \end{cases}$$

The value of ξ is generally defined on $(-\infty, \infty)$. Persons with negative threshold values always satisfy the first rule thereby being prone to the possibility of accepting the information. For mathematical convenience, let us define the set of threshold values $\Xi(P)$ satisfying $\xi \leq Q(P)$ as $\Xi(P) := \{\xi \mid \xi \leq Q(P)\}$ and the complementary set of $\Xi(P)$ as $\overline{\Xi}(P) := \{\xi \mid \xi > Q(P)\}$.

Now, we consider the cumulative distribution function (CDF) $F(x)$ of the threshold value ξ in the population such that

$$F(x) = \int_{-\infty}^{x} f(\xi) d\xi$$

where $f(\xi)$ is the frequency distribution function (FDF) of the threshold value ξ in the population, such that $f(\xi)\Delta\xi$ with sufficiently small $\Delta\xi > 0$ corresponds to the frequency of individuals with the threshold in the range $[\xi, \xi + \Delta\xi]$ within the population. The value $F(x)$ means the frequency of individuals with the threshold value ξ not beyond x within the population. The functions F and f are assumed to satisfy the following conditions:

- F and f are independent of time t;
- $f(\xi)$ is non-negative and integrable for any $\xi \in \mathbb{R}$;
- $F(x)$ is non-negative, non-decreasing, and continuous for any $x \in \mathbb{R}$;
- $\lim_{x\to\infty} F(x) = \int_{-\infty}^{\infty} f(\xi)d\xi = 1$ and $\lim_{x\to-\infty} F(x) = 0$;
- $\lim_{\xi\to\infty} f(\xi) = 0$ and $\lim_{\xi\to-\infty} f(\xi) = 0$.

The frequency of knowers $P(t)$ in the population at time t is described as

$$P(t) = \int_{-\infty}^{\infty} p(\xi, t)d\xi,$$

where $p(\xi, t)$ is the FDF of knower's threshold value ξ at time t in the population, such that $p(\xi, t)\Delta\xi$ with sufficiently small $\Delta\xi > 0$ corresponds to the frequency of knowers with the threshold in the range $[\xi, \xi + \Delta\xi]$ at time t in the population. For mathematical convention, we define the frequency of non-knowers $U(t) = 1 - P(t)$ in the population at time t as

$$U(t) = \int_{-\infty}^{\infty} u(\xi, t)d\xi,$$

where $u(\xi, t) = f(\xi) - p(\xi, t)$ means the FDF of non-knowers' threshold values ξ at t.

The transition probability that the non-knower with the threshold value ξ gets the information and changes to a knower in $[t, t + \Delta t]$ with sufficiently small $\Delta t > 0$ is now denoted by $\mathcal{B}(\xi, P)\Delta t$. In our modeling, $\mathcal{B}(\xi, P)$ is defined by

$$\mathcal{B}(\xi, P) = \begin{cases} B(P), & \xi \in \Xi(P); \\ 0, & \xi \in \overline{\Xi}(P), \end{cases} \tag{2.1}$$

where $B(P)$ is the coefficient of information acceptance for the non-knower with the threshold value of $\Xi(P)$ with $B(0) = 0$, $B(P) > 0$ for $P \in [0, 1]$.

2.3 Temporal change of the non-knower and knower frequencies

From the above setup, we can immediately get the following equation

$$u(\xi, t + \Delta t)\Delta\xi - u(\xi, t)\Delta\xi = -\mathcal{B}(\xi, P(t))\Delta t \cdot u(\xi, t)\Delta\xi, \tag{2.2}$$

where the left side corresponds to the change of the frequency of non-knowers during $[t, t + \Delta t]$ with sufficiently small $\Delta t > 0$ and the threshold value in the range $[\xi, \xi + \Delta\xi]$ with sufficiently small $\Delta\xi$. It means the number of non-knowers becoming knowers by accepting the information is equal to the right hand side given the expected reduction of the non-knower frequency by the transition probability defined above.

From the equation (2.2), we can derive the following equations as $\Delta t \to 0$:

$$\frac{\partial u(\xi, t)}{\partial t} = -\mathcal{B}(\xi, P(t))u(\xi, t).$$

Therefore, integrating both sides in terms of ξ over $\mathbb{R}$, we have

$$\frac{d}{dt}\int_{-\infty}^{\infty} u(\xi, t)d\xi = \frac{dU(t)}{dt} = -\int_{-\infty}^{\infty} \mathcal{B}(\xi, P(t))u(\xi, t)d\xi = -B(P(t))\int_{\Xi(P(t))} u(\xi, t)d\xi,$$

where we use the transition probability defined by (2.1). We have the following equation for the temporal change of the knower frequency $P(t)$:

$$\frac{dP(t)}{dt} = -\frac{dU(t)}{dt} = B(P(t))\int_{\Xi(P(t))} u(\xi, t)d\xi. \tag{2.3}$$

2.4 Initial condition for the population dynamics

For the initial condition at $t = 0$, we assume a portion of knowers within the population who play the role of initial transmitters of information. These initial knowers are given independently of their threshold values without bothering about how they become knowers. We give the initial distribution of the knower frequency by $p(\xi, 0)\Delta\xi = \theta(\xi)f(\xi)\Delta\xi$, where $\theta(\xi)$ determines the ratio of initial knowers in the subpopulation with the threshold value ξ such that $0 \le \theta(\xi) \le 1$. Since $p(\xi, 0)\Delta\xi + u(\xi, 0)\Delta\xi = f(\xi)\Delta\xi$, we have the following equation: $u(\xi, 0)\Delta\xi = \{1 - \theta(\xi)\}f(\xi)\Delta\xi$. Hence, we have

$$P(0) = \int_{-\infty}^{\infty} \theta(\xi)f(\xi)d\xi; \quad U(0) = \int_{-\infty}^{\infty} \{1 - \theta(\xi)\}f(\xi)d\xi = 1 - P(0). \tag{2.4}$$

2.5 Non-knower frequency of $\overline{\Xi}(P)$

There are non-knowers who have threshold values beyond the value of Q at time t. Their frequency is given by

$$U(t) - \int_{\Xi(P(t))} u(\xi,t)d\xi = \int_{\overline{\Xi}(P(t))} u(\xi,t)d\xi. \tag{2.5}$$

Since $P(t)$ is non-decreasing in time and $Q(P)$ is non-decreasing in terms of P, we note that the set $\overline{\Xi}(P(t))$ identifies all non-knowers who have not experienced any moment at which the value of $Q(P(t))$ is more than the threshold value until time t. Therefore, the non-knowers belonging to the above integral is only those who remain at the non-knower state from the initial time to time t. That is, $u(\xi,t) = u(\xi,0)$ for $\xi \in \overline{\Xi}(P(t))$ so that

$$\int_{\overline{\Xi}(P(t))} u(\xi,t)d\xi = \int_{\overline{\Xi}(P(t))} u(\xi,0)d\xi = \int_{\overline{\Xi}(P(t))} \{1-\theta(\xi)\}f(\xi)d\xi. \tag{2.6}$$

2.6 Closed equation for the knower frequency

From (2.3), (2.5) and (2.6) in the preceding modeling arguments, we have

$$\begin{aligned}\frac{dP(t)}{dt} &= B(P(t))\int_{\Xi(P(t))} u(\xi,t)d\xi = B(P(t))\left[U(t) - \int_{\overline{\Xi}(P(t))} u(\xi,t)d\xi\right]\\ &= B(P(t))\left[U(t) - \int_{\overline{\Xi}(P(t))}\{1-\theta(\xi)\}f(\xi)d\xi\right] = B(P(t))\left[1 - P(t) - \int_{\overline{\Xi}(P(t))}\{1-\theta(\xi)\}f(\xi)d\xi\right].\end{aligned} \tag{2.7}$$

The equation is closed in terms of $P(t)$ and it can be regarded as an autonomous ordinary differential equation to describe the temporal change of knower frequency within the population. To guarantee the reasonableness of (2.7), we have the following theorem:

Theorem 2.1. *For any $P(0)$ such that $0 \le P(0) \le 1$, it is true that $P(0) \le P(t) \le 1$ for any $t > 0$.*

Proof. If $P = 0$, then $B(0) = 0$ and we have $|\int_{\overline{\Xi}(0)}\{1-\theta(\xi)\}f(\xi)d\xi| < \infty$, so from (2.7), $\frac{dP(t)}{dt}|_{P=0} = 0$. As such, $P \equiv 0$ is a solution for (2.7). From the uniqueness of solution for (2.7), if $P(0) = 0$, then $P(t) = 0$ for all $t > 0$. If $P(0) > 0$, then $P(t) > 0$ for all $t > 0$ since $P(t)$ is non decreasing in time. More so,

$$\left.\frac{dP(t)}{dt}\right|_{P=1} = -B(1)\int_{\overline{\Xi}(1)}\{1-\theta(\xi)\}f(\xi)d\xi \le 0.$$

Further, we formally have

$$\left.\frac{dP(t)}{dt}\right|_{P>1} = B(P)\left[1 - P - \int_{\overline{\Xi}(P)}\{1-\theta(\xi)\}f(\xi)d\xi\right] < 0.$$

Therefore, it is mathematically impossible that P goes beyond 1 in any finite time from the initial value $P(0) \le 1$. Besides, for $P(0) > 0$, we have

$$\begin{aligned}\left.\frac{dP(t)}{dt}\right|_{t=0} &= B(P(0))\left[1 - P(0) - \int_{\overline{\Xi}(P(0))}\{1-\theta(\xi)\}f(\xi)d\xi\right]\\ &= B(P(0))\left[1 - \int_{-\infty}^{\infty}\theta(\xi)f(\xi)d\xi - \int_{\overline{\Xi}(P(0))}\{1-\theta(\xi)\}f(\xi)d\xi\right]\\ &= B(P(0))\left[1 - \int_{\Xi(P(0))}\theta(\xi)f(\xi)d\xi - \int_{\overline{\Xi}(P(0))}f(\xi)d\xi\right]\\ &> B(P(0))\left[1 - \int_{\Xi(P(0))}f(\xi)d\xi - \int_{\overline{\Xi}(P(0))}f(\xi)d\xi\right] = B(P(0))\left[1 - \int_{-\infty}^{\infty}f(\xi)d\xi\right] = 0,\end{aligned}$$

so that P must increase at $t = 0$, and $P(t) > P(0)$ for all $t > 0$ with $P(0) > 0$. These results establish the invariance of $P(t)$ such that $P(0) \le P(t) \le 1$ for all $t > 0$. □

The formula of the above equation depends on the limit $\lim_{P\to1} Q(P)$ which is now formally equal to $\sup_P Q(P)$ because the function of $Q(P)$ is assumed to be non-decreasing in terms of P. If $\lim_{P\to1} Q(P) < \infty$, that is, if $Q(1)$ is a finite value, the set $\overline{\Xi}(P)$ becomes empty for a certain value of $P = P_c \le 1$, when $f(\xi) = 0$ for any $\xi > \xi_m$ with a finite value ξ_m, and $Q(P_c) \ge \xi_m$. In such a case, the integral in (2.7) necessarily becomes zero for any $\xi \ge Q(P_c)$ because $f(\xi) = 0$. Therefore, in this case, we can express (2.7) as

$$\frac{dP(t)}{dt} = B(P(t))G(P(t)), \tag{2.8}$$

where

$$G(P) = \begin{cases} 1 - P - \int_{Q(P)}^{\xi_m} \{1 - \theta(\xi)\} f(\xi) d\xi, & Q(P) < \xi_m; \\ 1 - P, & Q(P) \geq \xi_m. \end{cases} \tag{2.9}$$

2.7 A simple setup for the initial knowers and the strength of social recognition effect

In the subsequent sections, we shall mathematically analyze our population dynamics model for information spread, introducing some specific distributions of threshold value in the population. For mathematical simplicity, we assume that the initial knowers are chosen at random with probability θ_0 $(0 < \theta_0 < 1)$ independent of each individual's threshold value so that $\theta(\xi) = \theta_0$. Then, from (2.4), we have $P(0) = \theta_0$.

In addition, let us consider the strength of social recognition effect proportional to the frequency of knowers within the population. That is, we introduce $Q(P) = \alpha P$ with a positive constant α. This formula of $Q(P)$ is the simplest one satisfying the mathematical features given in Sect. 2.2. The parameter α reflects the sensitivity of the society to the spread of information. As α gets larger, the society is more sensitive to the spread of information.

3. The Case of Everywhere Positive Distribution

In this section, we consider the model with $f(\xi) > 0$ for any $\xi \in \mathbb{R}$ such that the equation (2.7) becomes

$$\frac{dP(t)}{dt} = \alpha P(t) G(P(t)), \tag{3.1}$$

where

$$G(P) := 1 - P - (1 - \theta_0) \int_{\alpha P}^{\infty} f(\xi) d\xi. \tag{3.2}$$

3.1 Existence of equilibrium states

As for the equilibrium state $P = P^* > 0$ where $dP(t)/dt = 0$, we have $G(P^*) = 0$. The function $G(P)$ defined by (3.2) is continuous, and we have

$$G(\theta_0) = (1 - \theta_0)\left[1 - \int_{\alpha P}^{\infty} f(\xi) d\xi\right] > 0; \quad G(1) = -(1 - \theta_0) \int_{\alpha}^{\infty} f(\xi) d\xi < 0.$$

So, there is at least one value of $P = P^*$ such that $\theta_0 < P^* < 1$. Hence, we have the following theorem:

Theorem 3.1. *There is at least one equilibrium state $P = P^*$ for (3.1) with (3.2) such that $\theta_0 < P^* < 1$.*

From the idea of standard local stability analysis, we can obtain the following result for the equilibrium state $P = P^* \in (\theta_0, 1)$.

Theorem 3.2. *The equilibrium state $P = P^*$ for (3.1) with (3.2) such that $\theta_0 < P^* < 1$ is locally asymptotically stable if $(1 - \theta_0)\alpha f(\alpha P^*) < 1$.*

Proof. From (3.1) with (3.2), we have $L(P) := dP/dt = \alpha P G(P)$ such that P^* is locally asymptotically stable if $dL/dP|_{P=P^*} < 0$. Now,

$$\left.\frac{dL(P)}{dP}\right|_{P=P^*} = \alpha G(P^*) + \alpha P^* \left.\frac{dG(P)}{dP}\right|_{P=P^*} = \alpha P^* \left.\frac{dG(P)}{dP}\right|_{P=P^*}.$$

$\frac{dL(P)}{dP}|_{P=P^*} < 0$ implies that $\frac{dG(P)}{dP}|_{P=P^*} < 0$ since $P^* > 0$. So, we have

$$\frac{dG(P)}{dP} = -1 - (1 - \theta_0) \frac{d}{dP} \int_{\alpha P}^{\infty} f(\xi) d\xi = -1 + (1 - \theta_0)\alpha f(\alpha P)$$

such that $\frac{dG(P)}{dP}|_{P=P^*} < 0$ results to $(1 - \theta_0)\alpha f(\alpha P^*) < 1$. This completes the proof for the establishment of sufficient condition for local stability. □

From this theorem, we see that even when a subunity equilibrium state exists, it may be unstable. In the following sections, we investigate the detail of the dynamical nature of the model (3.1) with (3.2) according to a given formula of the distribution $f(\xi)$ everywhere positive for $\xi \in \mathbb{R}$.

3.2 A specific distribution

We consider the threshold distribution defined as

$$f(\xi)=\frac{1}{\sigma\sqrt{2}}\exp\left(-\sqrt{2}\frac{|\xi-\bar{\xi}|}{\sigma}\right)=\begin{cases}\dfrac{1}{\sigma\sqrt{2}}\exp\left(\sqrt{2}\dfrac{|\xi-\bar{\xi}|}{\sigma}\right), & \xi<\bar{\xi};\\ \dfrac{1}{\sigma\sqrt{2}}\exp\left(-\sqrt{2}\dfrac{|\xi-\bar{\xi}|}{\sigma}\right), & \xi\ge\bar{\xi},\end{cases} \tag{3.3}$$

with mean $\bar{\xi}$ and variance σ^2 (Fig. 1). Following Theorem 3.1, there is at least one equilibrium state $P=P^*$ such that $0<P^*<1$. In terms of (3.2), we now have

$$G(P)=\begin{cases}\theta_0-P+\dfrac{1}{2}(1-\theta_0)\exp\left[\sqrt{2}\dfrac{\alpha}{\sigma}\left(P-\dfrac{\bar{\xi}}{\alpha}\right)\right], & P<\dfrac{\bar{\xi}}{\alpha};\\ 1-P-\dfrac{1}{2}(1-\theta_0)\exp\left[-\sqrt{2}\dfrac{\alpha}{\sigma}\left(P-\dfrac{\bar{\xi}}{\alpha}\right)\right], & P\ge\dfrac{\bar{\xi}}{\alpha}.\end{cases} \tag{3.4}$$

$G(P)$ is continuous in $[0,1]$ as seen in Fig. 2 since

$$\lim_{P\to\frac{\bar{\xi}}{\alpha}-0} G(P)=\lim_{P\to\frac{\bar{\xi}}{\alpha}+0} G(P)=G\left(\frac{\bar{\xi}}{\alpha}\right)=\frac{1}{2}(1+\theta_0)-\frac{\bar{\xi}}{\alpha}. \tag{3.5}$$

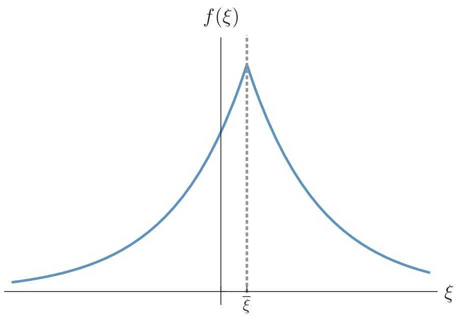

Fig. 1. A specific distribution function $f(\xi)$ of the threshold value ξ for the social recognition effect given by (3.3).

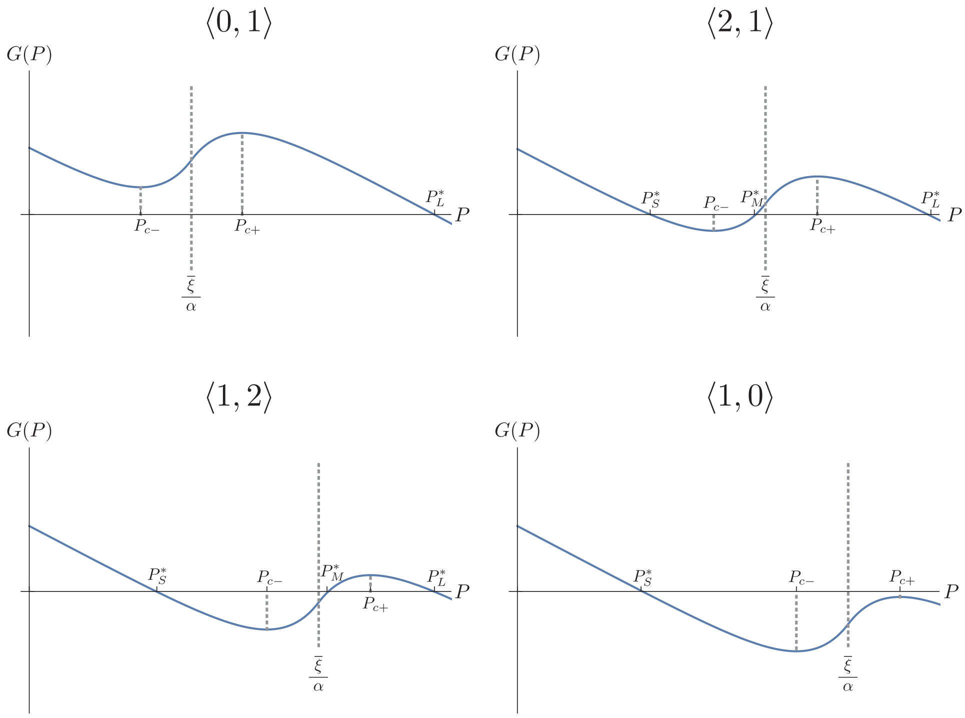

Fig. 2. Four possible cases of the graph of (3.4) corresponding to the conditions given as (3.6)–(3.11). In each figure, $\langle i,j\rangle$ represents the pair of numbers of non-trivial equilibrium states i and j in the intervals $(\theta_0,\bar{\xi}/\alpha)$ and $[\bar{\xi}/\alpha,1]$ respectively for the model (3.1) with (3.3). Intersections of $G(P)$ and the horizontal axis denoted by P_S^*, P_M^*, P_L^* are the non-trivial equilibrium states for the model (3.1) with (3.3).

Existence of equilibrium states

The symbol $\langle i, j\rangle$ in Fig. 2 represents the respective numbers of formal non-trivial equilibrium states in each of the intervals $(\theta_0, \bar{\xi}/\alpha)$ and $[\bar{\xi}/\alpha, 1]$. The total number of non-trivial equilibrium states in the complete interval $(\theta_0, 1]$ is given by $i + j$. We can get the following necessary and sufficient conditions for $\langle i, j\rangle$ based on the conditions obtained in Appendix A.

- For $\langle 0, 1\rangle$, we have either of

$$\frac{\sigma}{\alpha} < \frac{1}{\sqrt{2}}(1-\theta_0) \quad \text{and} \quad \frac{\bar{\xi}}{\alpha} < \theta_0 + \frac{\sigma}{\alpha\sqrt{2}}\left[1 + \ln\frac{1-\theta_0}{\sqrt{2}} - \ln\frac{\sigma}{\alpha}\right] \quad \text{or} \tag{3.6}$$

$$\frac{\sigma}{\alpha} \geq \frac{1}{\sqrt{2}}(1-\theta_0) \quad \text{and} \quad \frac{\bar{\xi}}{\alpha} < \frac{1}{2}(1+\theta_0). \tag{3.7}$$

- For $\langle 2, 1\rangle$, we have

$$\frac{\sigma}{\alpha} < \frac{1}{\sqrt{2}}(1-\theta_0) \quad \text{and} \quad \theta_0 + \frac{\sigma}{\alpha\sqrt{2}}\left[1 + \ln\frac{1-\theta_0}{\sqrt{2}} - \ln\frac{\sigma}{\alpha}\right] < \frac{\bar{\xi}}{\alpha} < \frac{1}{2}(1+\theta_0). \tag{3.8}$$

- For $\langle 1, 2\rangle$, we have

$$\frac{\sigma}{\alpha} < \frac{1}{\sqrt{2}}(1-\theta_0) \quad \text{and} \quad \frac{1}{2}(1+\theta_0) < \frac{\bar{\xi}}{\alpha} < 1 - \frac{\sigma}{\alpha\sqrt{2}}\left[1 + \ln\frac{1-\theta_0}{\sqrt{2}} - \ln\frac{\sigma}{\alpha}\right]. \tag{3.9}$$

- For $\langle 1, 0\rangle$, we have either of

$$\frac{\sigma}{\alpha} < \frac{1}{\sqrt{2}}(1-\theta_0) \quad \text{and} \quad \frac{\bar{\xi}}{\alpha} > 1 - \frac{\sigma}{\alpha\sqrt{2}}\left[1 + \ln\frac{1-\theta_0}{\sqrt{2}} - \ln\frac{\sigma}{\alpha}\right] \quad \text{or} \tag{3.10}$$

$$\frac{\sigma}{\alpha} \geq \frac{1}{\sqrt{2}}(1-\theta_0) \quad \text{and} \quad \frac{\bar{\xi}}{\alpha} > \frac{1}{2}(1+\theta_0). \tag{3.11}$$

Stability of equilibrium states

Based on Theorem 3.2, we can obtain the sufficient condition for local asymptotic stability of the equilibrium state $P = P^*$ for the model (3.1) with (3.3). For $P^* < \bar{\xi}/\alpha$, it is locally asymptotically stable if

$$P^* < P_{c-} := \frac{\bar{\xi}}{\alpha} - \frac{\sigma}{\alpha\sqrt{2}}\ln\frac{\alpha}{\sigma\sqrt{2}}(1-\theta_0),$$

and for $P^* \geq \bar{\xi}/\alpha$, $P = P^*$ is locally asymptotically stable if

$$P^* > P_{c+} := \frac{\bar{\xi}}{\alpha} + \frac{\sigma}{\alpha\sqrt{2}}\ln\frac{\alpha}{\sigma\sqrt{2}}(1-\theta_0).$$

Furthermore, we can consider the global stability making use of the sign of (3.1) determined by that of $G(P)$. As shown in Fig. 2, we have non-trivial equilibrium states $\langle 0, 1\rangle$, $\langle 2, 1\rangle$, $\langle 1, 2\rangle$ and $\langle 1, 0\rangle$ in the intervals $(0, \bar{\xi}/\alpha)$ and $[\bar{\xi}/\alpha, 1]$ respectively. While the local stability of each equilibrium state can be obtained by applying Theorem 3.2, we can get the following result about the global stability based on the sign of $G(P)$ shown in Fig. 2.

Theorem 3.3. *When there is only one equilibrium state in the whole interval* $(\theta_0, 1]$*, it is always globally asymptotically stable. On the other hand, when there are three formal equilibrium states, the smallest and the largest are locally asymptotically stable while the middle one is unstable.*

Parameter dependence of the equilibrium value

With respect to $\bar{\xi}/\alpha$ as the bifurcation parameter of P^*, we have the critical value $\bar{\xi}_c/\alpha$ for a sufficiently small frequency of the initial knowers $P(0) = \theta_0$ as seen in Fig. 3 which shows there is a critical mean threshold value $\bar{\xi}_c$ for the population. Since $P(t)$ monotonically increases with time, a mean threshold value below $\bar{\xi}_c$ makes the system converge to the larger asymptotically stable equilibrium state while any one above $\bar{\xi}_c$ makes the system converge to the smaller asymptotically stable equilibrium state. This is understandable since the mean threshold value measures the acceptability/attractiveness of the information to the people. The proportion of knowers becomes drastically small when the community has a high mean threshold value thereby making the information highly unacceptable or ignored. A low mean threshold value indicates that the information is readily welcome in the society.

We have the critical value σ_c/α in Fig. 4 with σ/α as the bifurcation parameter of P^*. When such a critical variance exists, at a variance below it the system converges to the smaller equilibrium state. A variance above the critical variance drives the system to the larger equilibrium state. For sufficiently large mean of the threshold value, the

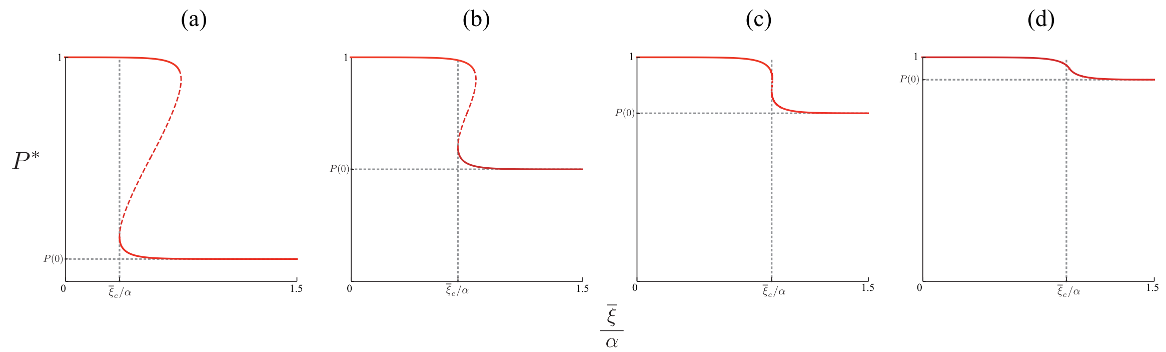

Fig. 3. Bifurcation diagram for P^* with parameter $\bar{\xi}/\alpha$ according to the model (3.1) with (3.3). (a) $P(0) = 0.10$, (b) $P(0) = 0.50$, (c) $P(0) = 0.75$, (d) $P(0) = 0.90$. Commonly, $\sigma/\alpha = 0.14$. $\bar{\xi}_c/\alpha$ is the critical value for $\bar{\xi}/\alpha$. For $\bar{\xi}/\alpha \geq \bar{\xi}_c/\alpha$, the frequency of knowers P necessarily approaches the equilibrium state P^* of the lowest branch. The critical value increases with the initial proportion of knowers.

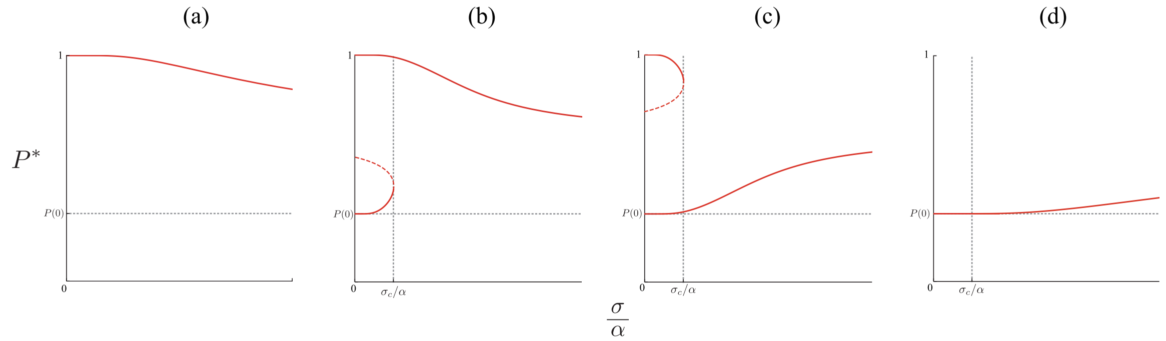

Fig. 4. Bifurcation diagram for P^* with parameter σ/α according to the model (3.1) with (3.3). (a) $\bar{\xi}/\alpha = 0.25$, (b) $\bar{\xi}/\alpha = 0.55$, (c) $\bar{\xi}/\alpha = 0.75$, (d) $\bar{\xi}/\alpha = 1.50$. Commonly, $P(0) = \theta_0 = 0.30$. In (b), for $\sigma/\alpha \leq \sigma_c/\alpha$, the frequency of knowers P necessarily approaches the equilibrium state P^* of the lowest branch. Also in (c), it necessarily approaches that of the lowest branch.

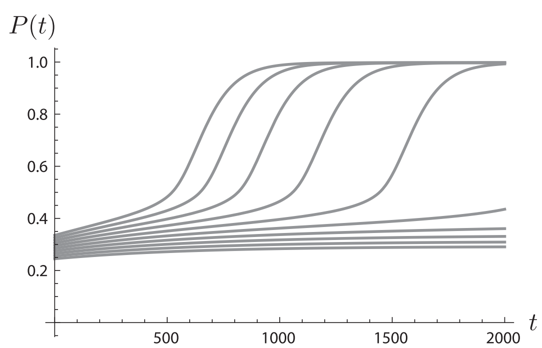

Fig. 5. Temporal variation of $P(t)$ given by the model (3.1) with (3.3) for different initial values $P(0) = \theta_0$. Commonly, $\alpha = 0.011$, $\sigma/\alpha = 0.14$ and $\bar{\xi}/\alpha = 0.50$. A slight difference in the initial proportion of knowers may cause a drastic difference on the consequence of information spread within the population.

reluctance of individuals to accept the information leads to the smaller equilibrium proportion of knowers [Figs. 4(c) and 4(d)]. In contrast, sufficiently small mean threshold value leads to the larger equilibrium proportion of knowers [Fig. 4(a)]. This implies that the larger proportion of knowers at the equilibrium state can be reached with sufficiently large proportion of individuals who have relatively small threshold values to accept the information, that is, relatively high gullibility for the information.

It should be noted that the equilibrium value P^* necessarily depends on the initial value $P(0) = \theta_0$, as seen in Fig. 5. The equilibrium state is uniquely determined by the initial condition. The bifurcation branches of P^* given in Fig. 6 with parameter θ_0 are obtained by first solving $G(P) = 0$ for θ_0. Then θ_c appears as the critical value for some small value of ξ. In case of Figs. 6(b) and 6(c), the system necessarily converges to the smaller equilibrium state for the initial value $P(0) = \theta_0$ below the critical value θ_c while it goes to the larger equilibrium state for the initial value beyond the critical value θ_c. As a whole, we see that the proportion of knowers gets large for a sufficiently large proportion of initial knowers while it becomes small for too small proportion of initial knowers.

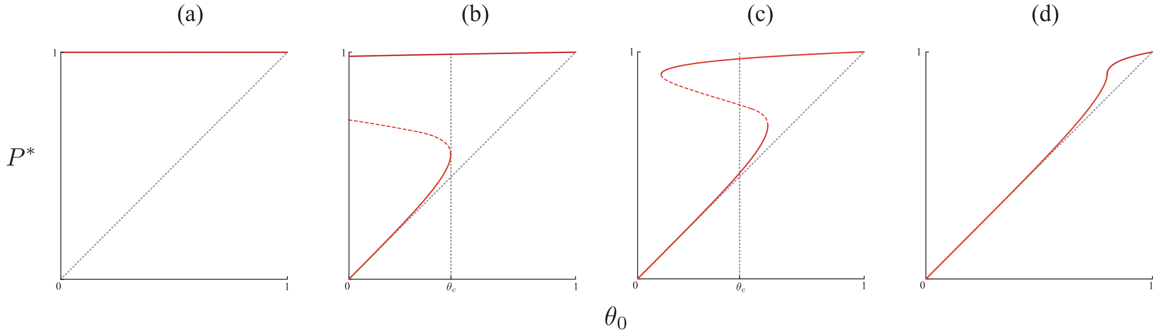

Fig. 6. Bifurcation diagram for P^* with the initial value $P(0) = \theta_0$ according to the model (3.1) with (3.3). (a) $0 < \bar{\xi}/\alpha < \bar{\xi}_*/\alpha$, (b) $1/2 < \bar{\xi}/\alpha < 1 - \bar{\xi}_*/\alpha$, (c) $\bar{\xi}/\alpha = 1 - \bar{\xi}_*/\alpha$, (d) $1 - \bar{\xi}_*/\alpha < \bar{\xi}/\alpha < 1$. Commonly, $\bar{\xi}_*/\alpha := (\sigma/\alpha\sqrt{2})(1 + \ln(\alpha/\sigma\sqrt{2}))$ and $\sigma/\alpha = 0.14$. θ_c is the critical value for θ_0. For $\theta_0 \leq \theta_c$, the frequency of knowers P necessarily approaches P^* of the lowest branch.

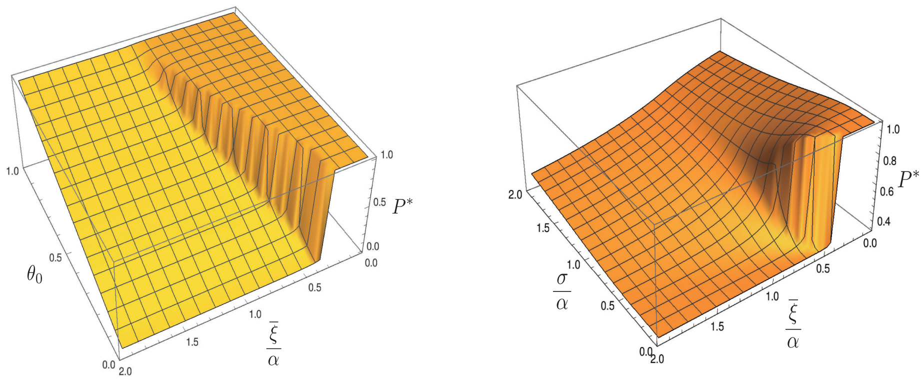

Fig. 7. Dependence of the equilibrium value P^* on $(\theta_0, \bar{\xi}/\alpha)$ with $\sigma/\alpha = 0.14$ and on $(\sigma/\alpha, \bar{\xi}/\alpha)$ with $\theta_0 = 0.3$ for the model (3.1) with (3.3). Commonly, $\alpha = 1.0$.

Further, we can find the following result about the equilibrium value of P:

Theorem 3.4. *The system converges to the equilibrium state at which the equilibrium value of P is greater than $\bar{\xi}/\alpha$ only in the case of $\langle 0, 1 \rangle$. In any other case, the equilibrium value of P is necessarily smaller than $\bar{\xi}/\alpha$.*

The proof of this theorem is found in Appendix B. Figure 7 shows that P converges to the smallest equilibrium state P_S^* when the mean threshold value is sufficiently large. For small mean threshold value, P converges to the largest equilibrium state P_L^*. Further, as we have seen, the criticality of the mean threshold value can appear only for sufficiently small variance (see also Figs. 3 and 4). Figure 8 numerically shows such criticality about the initial value (θ_0), the mean threshold value ($\bar{\xi}/\alpha$) and the variance (σ/α). It is implied that the consequence of information spread may significantly depend on the characteristics of a society and its relation to the spread of information represented by the parameters α, $\bar{\xi}$, and σ. More so, it may significantly depend on how the information begins spreading; for example, an idea may be propagated starting with a campaign strategy.

3.3 Normal distribution

As a typical choice of the everywhere positive distribution of thresholds, we may consider the normal distribution

$$f(\xi) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2} \left(\frac{\xi - \bar{\xi}}{\sigma} \right)^2 \right] \tag{3.12}$$

where σ is the standard deviation and $\bar{\xi}$ is the mean. As demonstrated by the numerical results in Fig. 9 in comparison with 8, the model (3.1) with normal distribution (3.12) has the same qualitative nature as the previous specific model with (3.3). It is hard to make a detailed mathematical analysis on the model with the normal distribution (3.12) as we did for the model with the previous specific everywhere positive distribution (3.3). Numerics about the model with the normal distribution (3.12) imply that its principal nature is qualitatively the same as the previous model with the specific distribution (3.3). We conjecture that the model with the everywhere positive distribution could have the same qualitative nature as the previous model as long as the distribution is unimodal. Actually, we will find a similar nature for a unimodal compact support distribution in a later section.

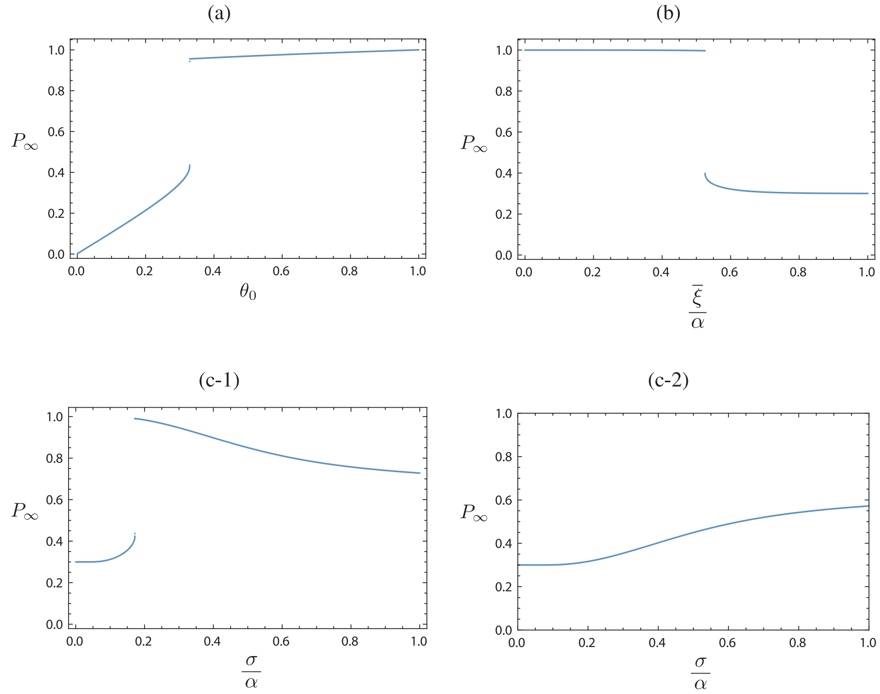

Fig. 8. Numerically obtained convergence of P for the model (3.1) with (3.3) and parameter (a) θ_0 with $\bar{\xi}/\alpha = 0.55$ and $\sigma/\alpha = 0.14$, (b) $\bar{\xi}/\alpha$ with $\theta_0 = 0.3$ and $\sigma/\alpha = 0.14$, (c-1) σ/α with $\theta_0 = 0.3$ and $\bar{\xi}/\alpha = 0.55$, (c-2) σ/α with $\theta_0 = 0.3$ and $\bar{\xi}/\alpha = 0.75$. Commonly, $\alpha = 1.0$.

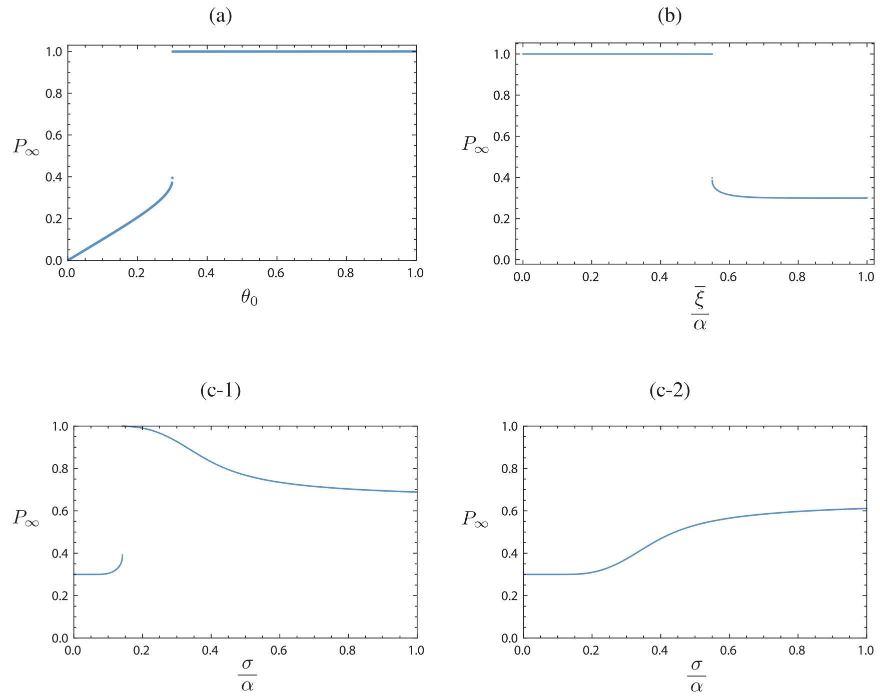

Fig. 9. Numerically obtained convergence of P for the model (3.1) with the normal distribution (3.12) and parameter (a) θ_0 with $\bar{\xi}/\alpha = 0.55$ and $\sigma/\alpha = 0.14$, (b) $\bar{\xi}/\alpha$ with $\theta_0 = 0.3$ and $\sigma/\alpha = 0.14$, (c-1) σ/α with $\theta_0 = 0.3$ and $\bar{\xi}/\alpha = 0.55$, (c-2) σ/α with $\theta_0 = 0.3$ and $\bar{\xi}/\alpha = 0.75$. Commonly, $\alpha = 1.0$. Compare with Fig. 8 for the model (3.1) with (3.3).

3.4 Monotonically decreasing distribution

In the previous section, we considered the continuous frequency distribution $f(\xi)$ defined on $\mathbb{R}$, where there exists some members belonging to any threshold range $[\xi_1, \xi_2] \subset \mathbb{R}$. In this section, we consider the model (3.1) with (3.2) according to another frequency distribution $f(\xi)$ defined on $\mathbb{R}_+ = [0, \infty)$ which is monotonically decreasing in terms of ξ:

$$f(\xi) = \begin{cases} 0, & \xi < 0; \\ h(\xi), & \xi \geq 0, \end{cases} \tag{3.13}$$

where the function $h(\xi)$ is monotonically decreasing in terms of ξ, and non-negative for any $\xi \in \mathbb{R}_+$, satisfying that $\sup_{\mathbb{R}_+} h(\xi) < \infty$, $\int_0^\infty h(\xi)\,d\xi = 1$, and $h(x) \to 0$ as $x \to \infty$.

For the model (3.1) with (3.2) according to (3.13), we can obtain the following general result which indicates that there is no bistable case for this model:

Theorem 3.5. *For the model (3.1) with (3.2) according to the monotonically decreasing frequency distribution (3.13) on $\mathbb{R}_+$, there always exists a globally asymptotical equilibrium state $P^* \in (\theta_0, 1]$ such that $P \to P^*$ as $t \to \infty$ for any parameter values.*

Proof. We have $G'(P) = -1 + (1 - \theta_0)\alpha f(\alpha P)$ for (3.2). Since the function $f(x)$ is monotonically decreasing for $x \in \mathbb{R}_+$, the sign of $G'(P)$ is always negative or changes only once from positive to negative as P increases. Besides, we have

$$G(\theta_0) = (1 - \theta_0)\left\{1 - \int_{\alpha\theta_0}^{\infty} f(\xi)\,d\xi\right\} > 0 \quad \text{and} \quad G(1) = -(1 - \theta_0)\int_{\alpha}^{\infty} f(\xi)\,d\xi \leq 0.$$

Hence, from the sign of $G'(P)$, we can find that the equation $G(P) = 0$ has a unique positive root $P^* \in (\theta_0, 1]$, and that $G(P) > 0$ for $P < P^*$ and $G(P) < 0$ for $P > P^*$. Since $dP/dt > 0$ for $P < P^*$ and $dP/dt < 0$ for $P > P^*$, we consequently get the theorem. □

4. The Case of Compact Support Distribution

In this section, we consider the compact support distribution $f(\xi)$ such that $f(\xi) = 0$ for $\xi \in (-\infty, 0] \cup [\xi_m, \infty)$. With the setup for $\theta(\xi)$ and $Q(P)$ in Sect. 2.7, the equation (2.8) is given with (2.9) which now becomes

$$G(P) = \begin{cases} 1 - P - (1 - \theta_0)\displaystyle\int_{\alpha P}^{\xi_m} f(\xi)\,d\xi & \text{for } P < \dfrac{\xi_m}{\alpha}; \\ 1 - P & \text{for } P \geq \dfrac{\xi_m}{\alpha}. \end{cases} \tag{4.1}$$

4.1 Uniform distribution

In this section, we consider the model (4.1) with the uniform distribution of ξ with $f(\xi)$ given as

$$f(\xi) = \begin{cases} 0, & \xi < 0; \\ \dfrac{1}{2\bar{\xi}}, & 0 \leq \xi \leq 2\bar{\xi}; \\ 0, & \xi > 2\bar{\xi}, \end{cases} \tag{4.2}$$

where $\bar{\xi}$ is the mean threshold value (Fig. 10). For $f(\xi)$ given by (4.2), the equation (2.9) is given with

$$G(P) = \begin{cases} \theta_0 - \left\{1 - \dfrac{\alpha}{2\bar{\xi}}(1 - \theta_0)\right\}P, & P \leq \dfrac{2\bar{\xi}}{\alpha}; \\ 1 - P, & P > \dfrac{2\bar{\xi}}{\alpha}. \end{cases} \tag{4.3}$$

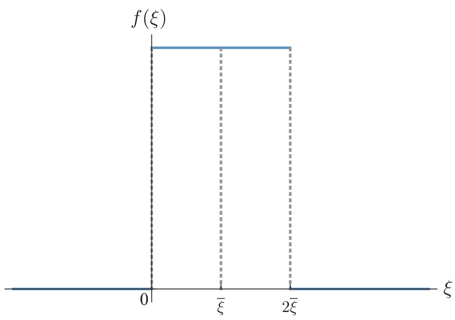

Fig. 10. The uniform frequency distribution function $f(\xi)$ of the threshold value ξ for the social recognition effect given by (4.2).

The equilibrium state in the interval $(0, 2\bar{\xi}/\alpha]$ is determined by $G(P) = 0$ such that

$$P = P^* = \frac{\theta_0}{1 - \frac{\alpha}{2\bar{\xi}}(1 - \theta_0)}, \tag{4.4}$$

while the equilibrium state in the interval $(2\bar{\xi}/\alpha, 1]$ is determined by $G(P) := 1 - P = 0$ and it is $P = P^* = 1$.

Then we obtain the following theorem:

Theorem 4.1. *The equation* (*2.8*) *with* (*4.3*) *has a unique equilibrium state which is globally asymptotically stable. When* $2\bar{\xi}/\alpha > 1$, *the equilibrium state* (*4.4*) *exists and is globally asymptotically stable. When* $2\bar{\xi}/\alpha \leq 1$, $P = P^* = 1$ *is globally asymptotically stable.*

Proof. When $2\bar{\xi}/\alpha > 1$, the equilibrium state (4.4) is always positive and less than 1. Since $G(0) = \theta_0 = P(0) < P^*$ and $G(P) > 0$ for any $P < P^*$, it is easy to find that P^* is globally asymptotically stable. It is clear that there is no other equilibrium state in $[0, 1]$ when $2\bar{\xi}/\alpha > 1$, since $P > 2\bar{\xi}/\alpha$ cannot occur for any $P \in [0, 1]$. When $2\bar{\xi}/\alpha \leq 1$, the equilibrium state (4.4) is non existent since it goes out of the range $(0, 2\bar{\xi}/\alpha]$. As such the equilibrium state is in $[2\bar{\xi}/\alpha, 1]$. $G(P)$ is positive for $P \in (0, 2\bar{\xi}/\alpha]$ and so is $G(P)$ for $P \in (2\bar{\xi}/\alpha, 1)$. Thus, $P(t)$ is monotonically increasing with time. Given that $2\bar{\xi}/\alpha \leq 1$ and that $P(t)$ is monotonically increasing with an upper bound $P = 1$, then we have the equilibrium state $P = P^* = 1$ which is globally asymptotically stable. Overall, we have a unique global equilibrium state for the system (2.8) with (4.3) depending on $2\bar{\xi}/\alpha$. □

The $\bar{\xi}/\alpha$-dependence of the equilibrium state P^* is shown in Fig. 11. We find that the larger $\bar{\xi}/\alpha$ leads to the smaller equilibrium state P^*. That is, a large mean threshold value or a weak sensitivity of society to the spread of a piece of information results in its less propagation. When $2\bar{\xi}/\alpha > 1$, the portion $1 - \alpha/(2\bar{\xi})$ of the population has threshold value beyond $Q(1) = \alpha$ which is the maximal strength of social recognition effect, and they never accept the information. Thus, for $\bar{\xi}/\alpha > 1$, the equilibrium state P^* must be less than one and decreasing in terms of $\bar{\xi}/\alpha$ as shown in Fig. 2. This may be due to a high level of education or the perceived lack of trustworthiness of the source of information. On the other hand, when $\bar{\xi}/\alpha \leq 1/2$, a relatively large number of individuals get to know the information in the long run due to their low threshold values of acceptance. This may be a result of gullibility on the part of the population or the reliability of the source.

4.2 Monotonically decreasing distribution

In this section, we consider the model (2.8) with (2.9) according to the following compact support frequency distribution $f(\xi)$ which is monotonically decreasing in terms of ξ:

$$f(\xi) = \begin{cases} 0, & \xi < 0; \\ h(\xi), & 0 \leq \xi < \xi_m; \\ 0, & \xi \geq \xi_m, \end{cases} \tag{4.5}$$

where the function $h(\xi)$ is monotonically decreasing in terms of ξ and positive for any $\xi \in [0, \xi_m)$, satisfying the conditions that $\sup_{[0,\xi_m)} h(\xi) < \infty$ and $\int_0^{\xi_m} h(\xi)\, d\xi = 1$.

For the model (2.8) with (2.9) according to (4.5), we can obtain the following general result:

Theorem 4.2. *For the model* (*2.8*) *with* (*2.9*) *according to the monotonically decreasing compact support frequency distribution* (*4.5*), *there always exists a globally asymptotical equilibrium state* $P^* \in (\theta_0, 1]$ *such that* $P \to P^*$ *as* $t \to \infty$ *for any parameter values.*

Proof. We have $G'(P) = -1 + (1 - \theta_0)\alpha h(\alpha P)$ for (2.9) in terms of $P < \xi_m/\alpha$. Since the function $h(x)$ is monotonically decreasing for $x \in [0, \xi_m)$, the sign of $G'(P)$ is always negative or changes only once from positive to negative as P increases. Besides, we have

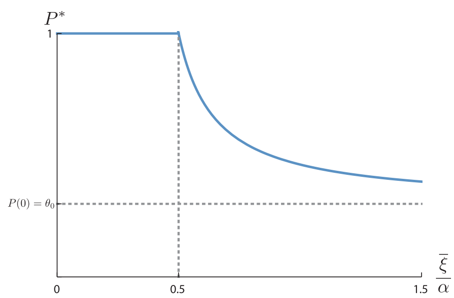

Fig. 11. $\bar{\xi}/\alpha$-dependence of the globally asymptotically stable equilibrium state P^* for the model (2.8) with (4.3) in the case of the uniformly distributed threshold value ξ given by the frequency distribution function (4.2).

$$G(\theta_0) = (1-\theta_0)\left\{1 - \int_{\alpha\theta_0}^{\xi_m} f(\xi)\,d\xi\right\} > 0, \quad G(1) = -(1-\theta_0)\int_{\alpha}^{\xi_m} f(\xi)\,d\xi \le 0,$$

and $G(\xi_m/\alpha) = 1 - \xi_m/\alpha$ as formal equations. Hence, from the sign of $G'(P)$ for $P \in (\theta_0, \min[1, \xi_m/\alpha])$, we can find that the equation $G(P) = 0$ has a unique positive root $P^* \in (\theta_0, \min[1, \xi_m/\alpha]]$, and that $G(P) > 0$ for $P < P^*$ and $G(P) < 0$ for $P > P^*$. Since $dP/dt > 0$ for $P < P^*$ and $dP/dt < 0$ for $P > P^*$, we consequently get the theorem. □

Therefore, there is no bistable case for this model. As a result, Theorem 4.2 could be regarded as correspondent to Theorem 3.5 in Sect. 3.4 for the model (3.1) with (3.2) according to the monotonically decreasing continuous frequency distribution (3.13) on $\mathbb{R}_+$.

4.3 Linearly increasing distribution

In this section, we consider the compact support linearly increasing distributions with a compact support $[0, \xi_m]$ for the frequency distribution $f(\xi)$ of the threshold value ξ for the social recognition effect (Fig. 12):

$$f(\xi) = \begin{cases} 0, & \xi < 0; \\ \dfrac{2}{\xi_m}\dfrac{\xi}{\xi_m}, & 0 \le \xi < \xi_m; \\ 0, & \xi \ge \xi_m, \end{cases} \tag{4.6}$$

where ξ_m is a positive constant. For this distribution, the mean threshold value $\bar{\xi}$ and variance σ^2 are given as $2\xi_m/3$ and $\xi_m^2/18$ respectively.

For the increasing linear distribution $f(\xi)$ given by (4.6), we can obtain the following result [Fig. 13(a)]:

Lemma 4.3. *With respect to the existence of equilibrium states in $(\theta_0, 1]$ for the equation (2.8) with (2.9) according to the increasing linear distribution (4.6), there are the following three cases:*

(i) There are only two different equilibrium states in $(\theta_0, \xi_m/\alpha)$ in addition to $P^* = 1$ if and only if the following condition is satisfied:

$$\sqrt{4\theta_0(1-\theta_0)} < \frac{\xi_m}{\alpha} < 1 \quad \textit{and} \quad \theta_0 < \frac{1}{2}.$$

(ii) There is only one equilibrium state in $(\theta_0, 1)$ if and only if $\xi_m/\alpha > 1$.

(iii) When any of these conditions (i) and (ii) is unsatisfied, there is no equilibrium state other than $P^* = 1$.

Proof. For $\alpha P \in [0, \xi_m)$, the function $G(P)$ given by (3.2) with (4.6) has two zeros if and only if $(\xi_m/\alpha)^2 - 4\theta_0(1-\theta_0) > 0$ because it is a quadratic function of P. Therefore, the equation $G(P) = 0$ has at most two roots in $(\theta_0, \min[\xi_m/\alpha, 1])$. Given that $\xi_m/\alpha < 1$ and the equation has two roots in $(\theta_0, \xi_m/\alpha)$, then (2.8) with (2.9) has three equilibrium states; that is, these two roots and $P^* = 1$. Since $G(\theta_0) > 0$, $G(\xi_m/\alpha) = 1 - \xi_m/\alpha$, and $G(1) = (1-\theta_0)\{(\alpha/\xi_m)^2 - 1\} > 0$ for $\xi_m/\alpha < 1$, we can find the condition for the number of roots of the equation $G(P) = 0$ in $(\theta_0, \min[\xi_m/\alpha, 1])$. This results in the lemma. It can be proven that there is no case where there is only one root for the equation $G(P) = 0$ in $(\theta_0, \xi_m/\alpha)$ when $\xi_m/\alpha < 1$. □

As a consequence of the analysis on the sign of $G(P)$ for the cases in Lemma 4.3, we can obtain the following theorem about the convergence of P as $t \to \infty$:

Theorem 4.4. *For the model (2.8) with (2.9) based on (4.6), a bistable situation occurs if and only if the condition (i) in Lemma 4.3 is satisfied.*

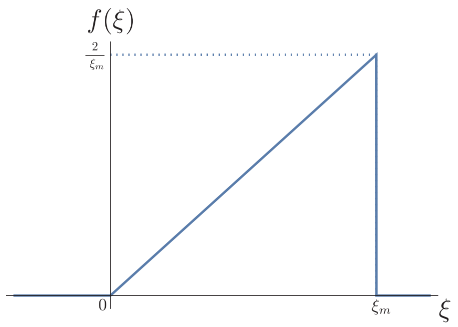

Fig. 12. The compact support linearly increasing frequency distributions $f(\xi)$ of the threshold value ξ for the social recognition effect given by (4.6).

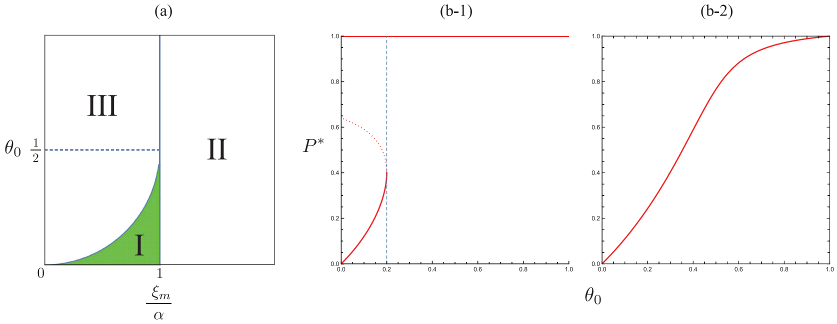

Fig. 13. (a) $(\xi_m/\alpha, \theta_0)$-dependence of the existence of positive equilibrium states $P^* \in (\theta_0, 1]$; (b-1, 2) Bifurcation diagram for P^* with the initial value $P(0) = \theta_0$ for the model (2.8) with (2.9) according to the increasing linear distribution (4.6). Numerically drawn with (b-1) $\xi_m/\alpha = 0.8$; (b-2) $\xi_m/\alpha = 1.05$. In (a), Regions I, II, and III, correspond to (i), (ii), and (iii) of Lemma 4.3 respectively.

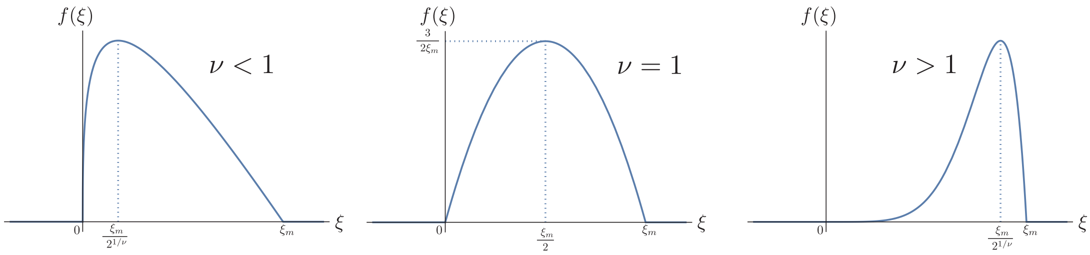

Fig. 14. The unimodal compact support frequency distribution $f(\xi)$ of the threshold value ξ for the social recognition effect given by (4.7).

The numerically drawn example of the bifurcation diagram for P^* with the initial value $P(0) = \theta_0$ in Fig. 13(b-1) clearly shows the existence of such a bistable situation. In contrast, Fig. 13(b-2) shows a case of (ii) in Lemma 4.3, which does not have any bistable situation with the parameter values belonging to Region II in Fig. 13(a). As indicated by Fig. 13(a), a bistable situation appears only when $\theta_0 < 1/2$ and $\xi_m/\alpha < 1$.

4.4 A specific unimodal distribution

Next, we consider the following specific unimodal distribution (Fig. 14):

$$f(\xi) = \begin{cases} 0, & \xi < 0; \\ \dfrac{(\nu+1)(2\nu+1)}{\nu\xi_m}\left(\dfrac{\xi}{\xi_m}\right)^{\nu}\left[1-\left(\dfrac{\xi}{\xi_m}\right)^{\nu}\right], & 0 \le \xi < \xi_m; \\ 0, & \xi \ge \xi_m, \end{cases} \tag{4.7}$$

where ξ_m and ν are positive constants. The mean threshold value $\overline{\xi}$ and variance σ^2 are given as $(2\nu+1)\xi_m/(2\nu+2)$ and $(2\nu+1)(5\nu+7)\xi_m^2/\{(\nu+3)(2\nu+2)^2(2\nu+3)\}$ respectively. For this distribution $f(\xi)$, the equation (2.8) is given with

$$G(P) = \begin{cases} \theta_0 - P + \dfrac{2\nu+1}{\nu}(1-\theta_0)\left(\dfrac{\alpha P}{\xi_m}\right)^{2\nu+1} - \dfrac{\nu+1}{\nu}(1-\theta_0)\left(\dfrac{\alpha P}{\xi_m}\right)^{2\nu+1}, & P < \dfrac{\xi_m}{\alpha}; \\ 1 - P, \quad P \ge \dfrac{\xi_m}{\alpha}. \end{cases} \tag{4.8}$$

By studying the nature of the function $G(P)$, we can get the following result:

Lemma 4.5. *With respect to the existence of equilibrium states in $(\theta_0, 1]$ for the equation (2.8) with (4.8), we have the following four cases:*

(i) There are only three different equilibrium states in $(\theta_0, 1)$ if and only if the following condition is satisfied:

$$1 \le \frac{\xi_m}{\alpha} < \frac{(1-\theta_0)(2\nu+1)(\nu+1)}{4\nu}, \quad G(P_{c-})G(P_{c+}) < 0 \quad \text{and} \quad P_{c+} < 1,$$

where

$$P_{c\pm} := \frac{\xi_m}{2\alpha}\left[1 - \sqrt{1 \pm \frac{4\nu}{(1-\theta_0)(2\nu+1)(\nu+1)}\frac{\xi_m}{\alpha}}\,\right]^{1/\nu}. \tag{4.9}$$

(ii) There are only two different equilibrium states in $(\theta_0, 1)$ *in addition to* $P^* = 1$ *if and only if*

$$\frac{\xi_m}{\alpha} < \min\left[1, \frac{(1-\theta_0)(2\nu+1)(\nu+1)}{4\nu}\right] \quad \text{and} \quad G(P_{c-})G(P_{c+}) < 0,$$

(iii) There is only one equilibrium state in $(\theta_0, 1)$ *if and only if*

$$1 \le \frac{\xi_m}{\alpha} < \frac{(1-\theta_0)(2\nu+1)(\nu+1)}{4\nu},$$

and $G(P_{c-})G(P_{c+}) < 0$ *with* $P_{c+} > 1$ *or* $G(P_{c-})G(P_{c+}) \ge 0$.

(iv) When any of these conditions (i), (ii) and (iii) is unsatisfied, there is no equilibrium state other than $P^* = 1$.

Proof. It is easily found that the function $G(P)$ defined for $P \in (0, \xi_m/\alpha)$ by (4.8) has two extremal points at $P = P_{c\pm}$ defined by (4.9) if and only if

$$\frac{\xi_m}{\alpha} < \frac{(1-\theta_0)(2\nu+1)(\nu+1)}{4\nu}, \tag{4.10}$$

and the shape of the curve of the function $G(P)$ defined for $P \in (0, \xi_m/\alpha)$ becomes similar with that of (3.4) in Sect. 3.2. When the condition (4.10) is not satisfied, the function $G(P)$ defined for $P \in (0, \xi_m/\alpha)$ is monotonically decreasing in terms of P. Since the root of $G(P) = 0$ in $(\theta_0, 1]$ could give an equilibrium state for the equation (2.8) with (4.8), we must distinguish two cases where $\xi_m/\alpha > 1$ and $\xi_m/\alpha < 1$. Taking account of $G(\xi_m/\alpha) = 1 - \xi_m/\alpha$, $G(1) \le 0$, and $\theta_0 < P_{c-} < P_{c+} < \xi_m/\alpha$ for any ν with respect to the function $G(P)$ defined for $P \in (0, \xi_m/\alpha)$, we can find the conditions given in the lemma about how many roots exist for $G(P) = 0$ in $(\theta_0, \min[1, \xi_m/\alpha])$. □

Making use of Lemma 4.5, we can derive the result shown in Fig. 15 about the $(\xi_m/\alpha, \theta_0)$-dependence of the existence of positive equilibrium states $P^* \in (\theta_0, 1]$ for the equation (2.8) with (4.8). The cusp point on the boundary between Regions II and III is given as

$$\frac{\xi_m}{\alpha} = x_c := \frac{(2\nu+1)(\nu+1)}{4\nu[1+\nu/2^{(\nu+1)/\nu}]}; \quad \theta_0 = \theta_c := \frac{\nu}{1+\nu/2^{(\nu+1)/\nu}}. \tag{4.11}$$

This is because the cusp point is so critical that $P_{c-} = P_{c+} = 2^{-1/\nu}\xi_m/\alpha$. The critical value x_c is monotonically decreasing in terms of ν with $x_c \to 1$ as $\nu \to \infty$ while θ_c is monotonically increasing in terms of ν with $\theta_c \to 2$ as $\nu \to \infty$. If $\theta_0 > \theta_c$ or $\xi_m/\alpha > x_c$, the equation (2.8) with (4.8) necessarily has a unique equilibrium state less than or equal to one, independently of the other parameters including ν and θ_0.

Consequently, from the analysis on the number of equilibrium states in $(\theta_0, 1)$ for the equation (2.8) with (4.8) and the shape of the curve of the function $G(P)$ for $P \in (\theta_0, 1)$ and Lemma 4.5, we can get the following result on the convergence of P as $t \to \infty$:

Theorem 4.6. *For the model* (2.8) *with* (4.8)*, a bistable situation occurs if and only if the condition (i) or (ii) in Lemma 4.5 is satisfied.*

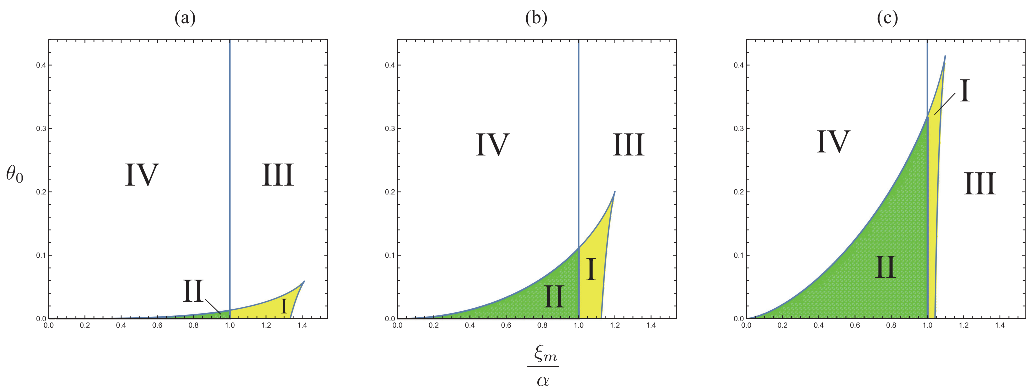

Fig. 15. $(\xi_m/\alpha, \theta_0)$-dependence of the existence of positive equilibrium states $P^* \in (\theta_0, 1]$ for the equation (2.8) with (4.8) by the unimodal compact support frequency distribution (4.7). Numerically drawn with the condition given in Lemma 4.5 for (a) $\nu = 0.5$; (b) $\nu = 1.0$; (c) $\nu = 2.0$. Regions I, II, III, and IV correspond to (i), (ii), (iii), and (iv) of Lemma 4.5 respectively.

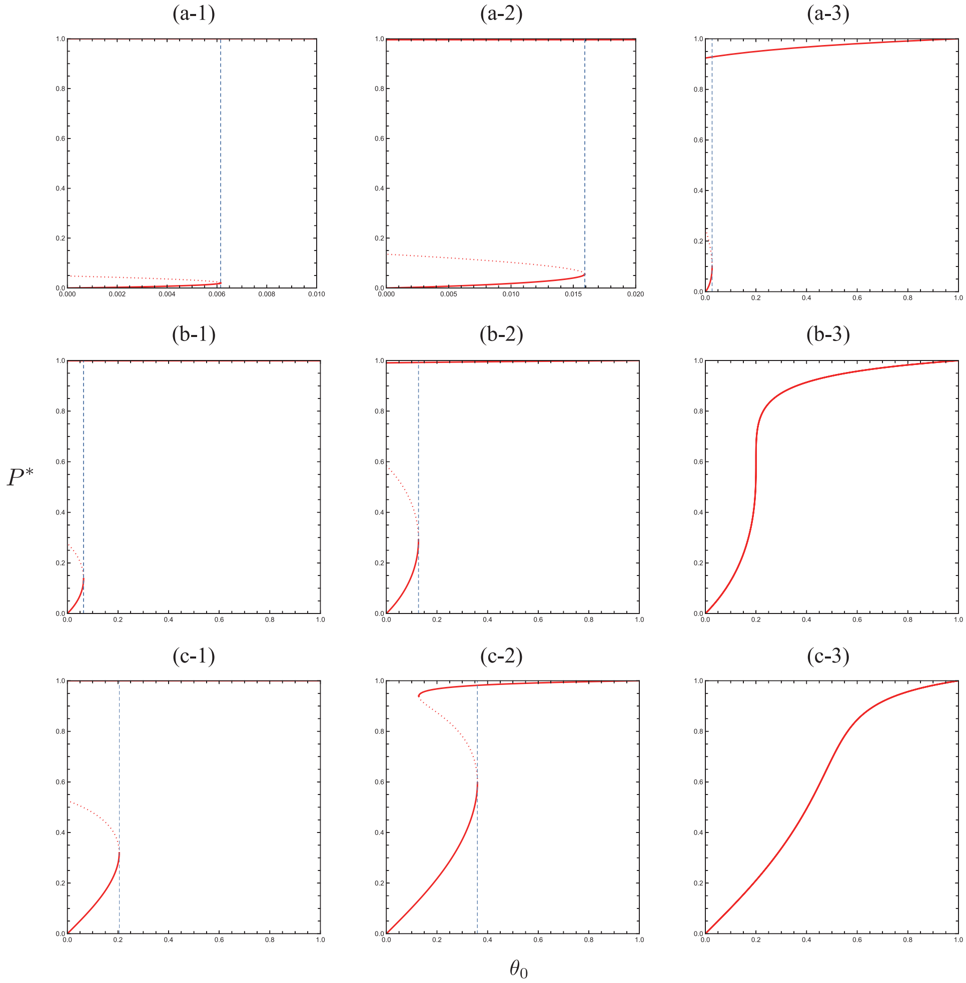

Fig. 16. Bifurcation diagrams for P^* with the initial value $P(0) = \theta_0$ according to the model (2.8) with (4.8). Dotted curves indicate unstable equilibria. Numerically drawn with (a-•) $\nu = 0.5$; (b-•) $\nu = 1.0$; (c-•) $\nu = 2.0$, and (•-1) $\xi_m/\alpha = 0.8$; (•-2) $\xi_m/\alpha = 1.05$; (•-3) $\xi_m/\alpha = 1.2$. For (a-1) and (a-2), the range of θ_0 is very close to zero.

Such a bistable situation must occur in a certain range of initial value θ_0 as indicated by Fig. 15. As already mentioned above, it is necessary for the appearance of such a bistable situation that $\theta_0 < \theta_c$ and $\xi_m/\alpha < x_c$ where θ_c and x_c are defined by (4.11). Although the ν-dependence is non-trivial, the numerical calculations in Fig. 15 imply that the parameter region for the formal bistable situation of Regions I and II becomes wider as ν gets larger while the existence and spatial configuration appear qualitatively same. As a result, such a bistable situation about the equilibrium states for the equation (2.8) with (4.8) becomes less observable for smaller ν. This tendency can be illustrated by the numerically drawn bifurcation diagrams for P^* in Fig. 16 too.

In conclusion, the results obtained in Sects. 3 and 4 imply that the unimodality of the frequency distribution could induce a bistable situation such that P converges to a small equilibrium value P^* from an initial value $P(0) = \theta_0$ less than a critical value while it converges to a distinctly different large equilibrium value P^* from an initial value greater than the critical value. In such a situation, the difference in the number of initial knowers may result in a significantly large difference in the final number of knowers in the population. In other words, the proportion of initial knowers determines the success of information spread within the population.

5. Discussion

The model with compact support uniform distribution tends to correspond to Granovetter's conceptual model. The behavior of the system is determined by the critical value of the mean threshold value $\bar{\xi}$ as indicated by Fig. 11. The model with everywhere positive distribution shows that the proportion of the population that ends up knowing an information largely depends on the strength of social recognition effect. A very large value of this effect on the population leads to the circulation of the information among a large proportion of the population in the long run. Conversely, a lower proportion of the population will end up knowing the information if the strength of the social recognition effect on the population is relatively small.

From the model with a specific distribution, the most important parameters are θ_0, $\bar{\xi}$ and σ. The proportion of initial knowers θ_0 represents those who have the task to carry out an initial operation to circulate a certain piece of information within a population. From our model, it is worthy of note that the final level of spread of a piece of information depends largely on the specific value of the initial proportion of knowers even when there is apparent bistability of equilibrium states.

Coincidentally, this proportion corresponds with the initial condition for our model. As such, it is both a parameter and an initial value from which the proportion of knowers continue to increase since we do not consider forgetfulness on the part of knowers. The initial proportion of knowers depends on the nature and situation of the piece of information. For instance, a single person can begin to spread a rumor; a syndicate may initiate a fake news; a new idea can begin with a pilot group within a population (e.g., the use of masks in preventing epidemics); top security secrets are known by very few people; and information from the mass media can be known initially by a large proportion of a population.

The mean threshold value of a population's social recognition effect, given as $\overline{\xi}$, characterizes how a community reacts to a specific kind of information. It is a kind of peak/mode behavior that is representative of the community. A large mean threshold indicates that a society is closed or conservative towards a particular kind of information, e.g., old people's attitude towards a hip hop concert. On the other hand, a small threshold mean shows that a society freely transmits a given piece of information. The standard deviation, σ, is a measure of the degree of scattering or variance of threshold values within the population. A small variance shows that the threshold values are similar among people while a large one implies that the threshold values highly vary within the population. In our analyses, $\overline{\xi}$ and σ describe the heterogeneity of the threshold values of individuals in the community.

For each of the parameters, there are critical values which determine whether a substantial proportion of the population gets to know the the piece of information or whether it is confined to an insignificant proportion.

The ordinary differential equation of P describing the threshold model is special for the reason that it explicitly depends on the initial value $P(0)$. This means that the temporal variation of proportion of knowers P explicitly depends on the initial condition. As such, the equilibrium state of the system is determined by the given initial value so that the equilibrium state varies for different initial values as seen in Figs. 5, 8(a), 9(a), 13(b-1), and 16.

The results show that each of the model parameters (initial value, mean threshold value for social response and standard deviation/variance) have critical values which determine the equilibrium state to which the system converges. However, the effect of variance is not as significant as that of the other two. From Figs. 8 and 9, it is seen that the proportion of knowers increases in terms of initial knowers and decreases in terms of mean threshold value of the social recognition effect. On the other hand, the effect of variance depends on the mean threshold value. For a relatively small mean threshold value, the proportion of knowers drastically increase once a critical variance is exceeded. A decrease of the proportion of knowers is then seen for large values of variance. For relatively large mean threshold value, the proportion of knowers rises continuously and peaks moderately.

Although we considered a model with a specific everywhere positive distribution, the normal distribution is the most popular. Even though the specific distribution may seem rather special, our numerical results imply that the results mathematically obtained from the model could be regarded as qualitatively the same as the results for the model with normal distribution. The model with normal distribution is difficult to analyze mathematically but we could carry out a detailed mathematical analysis on the model with the specific distribution. This is the reason why we analyzed the model with such a distribution.

We showed that the monotonically decreasing distribution of thresholds could not cause any bistable situation while bistability is possible for the unimodal distribution. In reality, the most observable distribution would be unimodal so that such a bistable situation could exist for a spread of information.

On the other hand, for a distribution with multiple peaks, the population dynamics about such an information spread could show some different features. However, that consideration is out of the scope of this paper. Although such a model would be certainly interesting from the mathematical point of view, it must be constructed with some reasonable assumptions. We may consider such a model in the future.

From the results discussed, it may be possible to control the initial proportion of knowers in order to achieve a desired purpose regarding the spread of information. On the other hand, individual threshold values and their distribution within a population can hardly be controlled. Such a control may only be possible under special conditions.

The analyses show that people can be stubborn in accepting a piece of information until a critical threshold value is reached. When the mean threshold value falls below the critical mean threshold value, there is a drastic increase in the frequency of knowers of the information due to an increasing level of sociability/acceptability. This scenario is commonly seen in the way people respond to most innovative ideas. Individuals always tend to resist potential changes to their ways of life but over time, with persistent awareness, they embrace change and the new idea becomes well circulated within their population.

Considering the importance of individual threshold values in accepting and diffusing information, the distribution of threshold values in a population is critical. We only chose to analyze the compact support and everywhere positive distributions of threshold values due to the fact that they are mathematically tractable. A more practical distribution is the normal distribution which is mathematically less tractable. Some numerical calculations with the normal distribution showed qualitatively similar results with the ones given in our analyses. Although it is likely that a specific distribution of threshold values causes some characteristic differences in the details of the analytical results, we think that our findings show the most principal nature of the model proposed in this paper. For example, if we consider the possibility of forgetting the obtained information in our modeling, this could lead to some new and interesting theoretical insights on the spread of information or some other matters transmissible in a society.

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Appendix A: The Existence of Non-trivial Equilibrium State for (3.1) with (3.2) According to (3.3)

Suppose there is an equilibrium state $P^* \in (\theta_0, \bar{\xi}/\alpha)$, it must satisfy $G(P) = 0$. We can easily see that $G(\theta_0) > 0$. When $\frac{\sigma}{\alpha} < \frac{1}{\sqrt{2}}(1-\theta_0)$, the function $G(P)$ is concave in terms of P with minimum point

$$P = P_{c-} := \frac{\bar{\xi}}{\alpha} - \frac{\sigma}{\alpha\sqrt{2}} \ln \frac{\alpha}{\sigma\sqrt{2}}(1-\theta_0) \tag{A·1}$$

so that $G(P_{c-}) = \theta_0 + \frac{\sigma}{\alpha\sqrt{2}}[1 + \ln \frac{\alpha}{\sigma\sqrt{2}}(1-\theta_0)] - \frac{\bar{\xi}}{\alpha}$. On the other hand, $G(P)$ is monotonically decreasing when $\frac{\sigma}{\alpha} \geq \frac{1}{\sqrt{2}}(1-\theta_0)$. $G(P_{c-}) = 0$ results to $\frac{\bar{\xi}_c}{\alpha} = \theta_0 + \frac{\sigma}{\alpha\sqrt{2}}[1 + \ln \frac{\alpha}{\sigma\sqrt{2}}(1-\theta_0)]$.

The equilibrium state $P^* \geq \bar{\xi}/\alpha$, if any exists, has to satisfy $G(P) = 0$. We see that $G(\theta_0) > 0$ and

$$G(1) = -\frac{1}{2}(1-\theta_0)\exp\left[-\sqrt{2}\frac{\alpha}{\sigma}\left(1-\frac{\bar{\xi}}{\alpha}\right)\right] < 0.$$

When $\frac{\sigma}{\alpha} < \frac{1}{\sqrt{2}}(1-\theta_0)$, the function $G(P)$ is convex in terms of P with maximum point at

$$P = P_{c+} := \frac{\bar{\xi}}{\alpha} + \frac{\sigma}{\alpha\sqrt{2}}\ln\frac{\alpha}{\sigma\sqrt{2}}(1-\theta_0) \tag{A·2}$$

and $G(P_{c+}) = 1 - \frac{\sigma}{\alpha\sqrt{2}}[1 + \ln\frac{\alpha}{\sigma\sqrt{2}}(1-\theta_0)] - \frac{\bar{\xi}}{\alpha}$. On the other hand, $G(P)$ is monotonically decreasing when $\frac{\sigma}{\alpha} \geq \frac{1}{\sqrt{2}}(1-\theta_0)$.

From Fig. 2, we have the following conditions for the existence of $\langle i, j\rangle$, where i and j are the numbers of equilibrium states in the intervals $(\theta_0, \bar{\xi}/\alpha)$ and $[\bar{\xi}/\alpha, 1]$ respectively. When $\frac{\sigma}{\alpha} < \frac{1}{\sqrt{2}}(1-\theta_0)$,

- $\langle 0, 1\rangle$: $G(P_{c-}) > 0$.
- $\langle 2, 1\rangle$: $G(P_{c-}) < 0$ and $G(\frac{\bar{\xi}}{\alpha}) > 0$.
- $\langle 1, 2\rangle$: $G(\frac{\bar{\xi}}{\alpha}) < 0$ and $G(P_{c+}) > 0$.
- $\langle 1, 0\rangle$: $G(P_{c+}) < 0$.

When $\frac{\sigma}{\alpha} \geq \frac{1}{\sqrt{2}}(1-\theta_0)$,

- $\langle 0, 1\rangle$: $G(\frac{\bar{\xi}}{\alpha}) > 0$.
- $\langle 1, 0\rangle$: $G(\frac{\bar{\xi}}{\alpha}) < 0$.

These equilibrium states agree with Theorem 3.1.

Appendix B: Proof for Theorem 3.4

For the case of $\langle 1, 2\rangle$, the second condition for its existence is given as expressed in the second part of (3.9). Since $\theta_0 < (1+\theta_0)/2$ for any $\theta_0 \in (0, 1)$, this condition requires that $\theta_0 < \bar{\xi}/\alpha$. This means that the case of $\langle 1, 2\rangle$ is only valid for the initial value satisfying that $\theta_0 < \bar{\xi}/\alpha$.

Next, for the case of $\langle 2, 1\rangle$, the second condition for its existence is expressed in the second part of (3.8). From the first condition for its existence, we have

$$\ln\frac{1-\theta_0}{\sqrt{2}} - \ln\frac{\sigma}{\alpha} > 0,$$

so that the second condition for its existence requires that $\theta_0 < \bar{\xi}/\alpha$ similar to the previous case. Therefore, the case of $\langle 2, 1\rangle$ is also only valid for the initial value satisfying that $\theta_0 < \bar{\xi}/\alpha$. Further, the first inequality of the second condition in (3.8) can be rewritten to be

$$\theta_0 + \frac{\sigma}{\alpha\sqrt{2}} < P_{c-}.$$

Hence we find the other necessary condition for the case of $\langle 2, 1\rangle$ that $\theta_0 < P_{c-}$. Now, since $G(\theta_0) = G(P(0)) > 0$, we can find that the value of P necessarily converges to the smallest equilibrium state P_S^* given by the smaller root of $G(P) = 0$ in both cases of $\langle 1, 2\rangle$ and $\langle 2, 1\rangle$.

In the case of $\langle 1, 2\rangle$ from Fig. 2, it is clear that P must converge to it since $P(0) = \theta_0 < \bar{\xi}/\alpha$ as shown in the above argument. Similarly, for the case of $\langle 2, 1\rangle$ in Fig. 2, since $P(0) = \theta_0 < P_{c-}$ as shown in the above argument, it is clear that P must converge to the smallest equilibrium state.